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An Analogue of the Thom Isomorphism for Crossed Products of a C^* Algebra by an Action of $\mathbb{R}$

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In this paper we show the existence and uniqueness of a natural isomorphism ϕ_α^j of $K_j(A)$ with $K_{j+1}(A \rtimes_\alpha \mathbb{R})$, $j \in \mathbb{Z}/2$, where $(A, \mathbb{R}, \alpha)$ is a C^* dynamical $\mathbb{R}$-system, K is the functor of topological K theory and $A \rtimes_\alpha \mathbb{R}$ is the crossed product of A by the action of $\mathbb{R}$. The Pimsner–Voiculescu exact sequence is obtained as a corollary. We show that given an α-invariant trace τ on A, with dual trace $\hat{\tau}$, one has $\hat{\tau}\phi_\alpha^1[u] = (1/2i\pi)\,\tau(\delta(u)u^*)$ for any unitary u in the domain of the derivation δ of A associated to the action α. Finally, we show that the crossed product of $C(S^3)$ (continuous functions on the 3 sphere) by a minimal diffeomorphism is a simple C^* algebra with no nontrivial idempotent.

INTRODUCTION

The aim of this paper is to prepare the ground for the statement of an index theorem for C^* dynamical systems (A, G, α) in the special case $G = \mathbb{R}^n$. For this we need to extend the construction of the Thom isomorphism (see, for instance, [1]) from its classical framework to the framework of C^* dynamical $\mathbb{R}$-systems. Our motivation for this is the following: in [4] we associated to each foliated manifold $(V, \mathscr{F})$ a C^* algebra $C^*(V, \mathscr{F})$ and an extension of $C^*(V, \mathscr{F})$: the C^* algebra $\mathscr{E}$ of pseudodifferential operators of order 0, yielding the exact sequence:

$$0 \to C^*(V, \mathscr{F}) \to \mathscr{E} \to C(S^*V) \to 0,$$

where S^*V is the unit sphere bundle of the tangent space to $\mathscr{F}$ (cf., [4, 9.1]). We showed that any holonomy invariant transverse measure Λ on the foliation defines a trace on $C^*(V, \mathscr{F})$ and hence an additive map $\dim_\Lambda$: $K_0(C^*(V, \mathscr{F})) \to \mathbb{R}$. In Section 8 of [4] we computed the composition of $\dim_\Lambda$ with the index map: $K^1(S^*V) \to K_0(C^*(V, \mathscr{F}))$, i.e., with the connecting map of the above exact sequence. The formula thus obtained extends the Atiyah–Singer index formula, the new ingredient being the cycle $[C] \in H(V, \mathbb{R})$ of the Ruelle–Sullivan current (of the transverse measure Λ).

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However, since transverse measures do not always exist it is very important to find a more primitive form of this index theorem, i.e., to compute, for any differential operator D elliptic along the leaves, the index $\operatorname{Ind} D \in K_0(C^*(V, \mathscr{F}))$ and not only $\dim_\Lambda(\operatorname{Ind} D)$. This problem does not have much sense unless one can first explicitly compute the enumerable abelian group $K_0(C^*(V, \mathscr{F}))$. Thanks to the remarkable breakthrough of Pimsner and Voiculescu [12] following earlier work of Cuntz [6], this computation was made in the simplest nontrivial case: the Kronecker foliation, by lines of irrational slope, in the two torus $\mathbb{R}^2/\mathbb{Z}^2$. As a corollary of the main result (Theorem 2 of Section IV) of the present paper we get that for flows without stable compact leaves the group $K_j(C^*(V, \mathscr{F}))$ is canonically isomorphic to $K^{j+1}(V)$ and that the image $\dim_\Lambda(K_0(C^*(V, \mathscr{F})))$ is equal to the image of $H^1(V, \mathbb{Z})$ by the Ruelle–Sullivan cycle $[C]$, and thus is equal to the range of the index map of [4]. The organisation of the paper is as follows: after easy preliminaries on C^* dynamical $\mathbb{R}$-systems (Section I) we show the existence and uniqueness of the Thom map ϕ_α^j: $K_j(A) \to K_{j+1}(A \rtimes_\alpha \mathbb{R})$ satisfying three very simple axioms. At the end of Section II we express the construction of ϕ_α^0 in an intrinsic way, in terms of connections and parallel transport. In Section III we compute ϕ_α^1 and prove that for any α-invariant trace τ on A the composition of ϕ_α^1 with the dual trace $\hat{\tau}$ is given, on any unitary $u \in A$ in the domain of the derivation δ, $\delta(u) = \lim_{t \to 0}(1/t)(\alpha_t(u) - u)$ by the equality:

$$\hat{\tau}(\phi_\alpha^1([u])) = \frac{1}{2i\pi} \tau(\delta(u)u^*).$$

The right-hand side was already known to have remarkable properties [2].

In Section IV we show that the Thom map ϕ_α^j is actually an isomorphism of $K_j(A)$ with $K_{j+1}(A \rtimes_\alpha \mathbb{R})$ and in Section V we derive the already-mentioned corollary together with the Pimsner–Voiculescu exact sequence, and a geometric construction of a simple C^* algebra without idempotent. In Section VI we gathered in an Appendix several technicalities which we did not desire to place in the text.

I. Preliminary on C^* Dynamical $\mathbb{R}$-Systems

By a C^* dynamical $\mathbb{R}$-system we mean a C^* algebra A together with a homomorphism $t \to \alpha_t \in \operatorname{Aut} A$ of $\mathbb{R}$ in the group of automorphisms of A, such that for each $x \in A$, the map $t \to \alpha_t(x)$ is norm continuous (cf. [11, 7.4.1]).

Given a C^* dynamical $\mathbb{R}$-system $(A, \mathbb{R}, \alpha)$, we let $\hat{A} = A \rtimes_\alpha \mathbb{R}$ be the crossed product of A by $\mathbb{R}$ (cf. [11, 7.6.5]). It contains as a dense * subalgebra the * algebra $L^1(\mathbb{R}, A)$ with product and involution defined by the equalities:

$$(a * b)(s) = \int a(t)\, \alpha_t b(s-t)\, dt \qquad \forall\, s \in \mathbb{R},$$

$$(a^*)(s) = \alpha_s(a(s^{-1})^*) \qquad \forall\, s \in \mathbb{R}.$$

We identify A with a subalgebra of the multipliers $M(A \rtimes_\alpha \mathbb{R})$ by associating to $x \in A$ the multiplier whose restriction to $L^1(\mathbb{R}, A)$ is

$$L_x(a)(s) = xa(s), \qquad (R_x(a))(s) = a(s)\, \alpha_s(x).$$

Finally the dual action, of $\hat{\mathbb{R}}$, on $A \rtimes_\alpha \mathbb{R}$ is such that

$$(\hat{\alpha}_s(a))(t) = \exp(its)\, a(t) \qquad \forall\, a \in L^1(\mathbb{R}, A).$$

The construction of the dual C^* dynamical system $(\hat{A}, \hat{\mathbb{R}}, \hat{\alpha})$ is functorial: if $\rho: A \to B$ is an equivariant homomorphism, the equality

$$(\hat{\rho}(a))(s) = \rho(a(s)) \qquad \forall\, a \in L^1(\mathbb{R}, A)$$

defines a homomorphism of $\hat{A}$ in $\hat{B}$ which is equivariant for the dual actions.

The duality of Takesaki–Takai (cf. [11, 7.9.3]) gives a canonical isomorphism of $\hat{\hat{A}}$ on $A \otimes k$, the tensor product of A by the C^* algebra k of compact operators in $L^2(\mathbb{R})$. Roughly speaking, it says that, up to the elementary operation of tensor product by k, the functor which to each C^* dynamical $\mathbb{R}$ system associates the dual one is its own inverse. In particular this functor is exact and we can state:

LEMMA 1. *Let $0 \to J \to^i A \to^\sigma B \to 0$ be an equivariant exact sequence, then the dual sequence $0 \to \hat{J} \to^{\hat{i}} \hat{A} \to^{\hat{\sigma}} B \to 0$ is also exact. If the first sequence is equivariantly split then the dual one is split.*

Proof. Clearly $\hat{\sigma} \circ \hat{i} = (\sigma \hat{\circ}\, i) = 0$, and if the inclusion $\operatorname{Ker} \hat{\sigma} \supset \operatorname{Im} \hat{i}$ was strict, the same would be true for the dual sequence $0 \to \hat{\hat{J}} \to^{\hat{\hat{i}}} \hat{\hat{A}} \to^{\hat{\hat{\sigma}}} \hat{\hat{B}} \to 0$, which is identical with

$$0 \longrightarrow J \otimes k \xrightarrow{i \otimes \mathrm{id}} A \otimes k \xrightarrow{\sigma \otimes \mathrm{id}} B \otimes k \longrightarrow 0.$$

For the second part, let r: $B \to A$ be an equivariant retraction, then $\hat{r}$ is a retraction $\hat{B} \to \hat{A}$. Q.E.D.

If A has no unit, let $\tilde{A}$ be the C^* algebra obtained by adjoining a unit to A, so $\tilde{A} = \{(x, \lambda), x \in A, \lambda \in \mathbb{C}\}$ with $(x, \lambda)\ (x', \lambda') = (xx' + \lambda' x + \lambda x', \lambda\lambda')$, $(x, \lambda)^* = (x^*, \bar{\lambda})$. The action α of $\mathbb{R}$ on A extends uniquely to an action $\tilde{\alpha}$ on

$\tilde{A}$, with $\tilde{\alpha}_t(x, \lambda) = (\alpha_t(x), \lambda)$ $\forall (x, \lambda) \in \tilde{A}$. If i_A, ε_A denote the equivariant homomorphisms $i_A(x) = (x, 0) \in \tilde{A}$ $\forall x \in A$, $\varepsilon_A(x, \lambda) = \lambda$ $\forall (x, \lambda) \in \tilde{A}$, we see from Lemma 1 that since the retraction $r_A(\lambda) = (0, \lambda) \in \tilde{A}$ is equivariant, the dual sequence $0 \to A \rtimes_\alpha \mathbb{R} \to \tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{R} \to \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \to 0$ is split. Let us state a corollary:

LEMMA 2. *With the above notations $(\hat{i}_A)_*$ is an isomorphism of $K_j(A \rtimes_\alpha \mathbb{R})$ on the kernel of $(\hat{\varepsilon}_A)_*$ in $K_j(\tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{R})$.*

Recall now that another action α' of $\mathbb{R}$ on A is called exterior equivalent to α iff there exists a unitary one cocycle $(u_t)_{t \in \mathbb{R}}$, $u_t \in M(A)$ $\forall t \in \mathbb{R}$ such that $t \to u_t x$ is norm continuous $\forall x \in A$ and that

$$\alpha'_t(x) = u_t \alpha_t(x) u_t^*.$$

Given such a cocycle one constructs an action β of $\mathbb{R}$ on $M_2(A) = M_2(\mathbb{C}) \otimes A$ by the equality (cf. [11, 8.11.2])

$$\beta_t \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} \alpha_t(x_{11}) & \alpha_t(x_{12}) u_t^* \\ u_t \alpha_t(x_{21}) & \alpha'_t(x_{22}) \end{bmatrix}.$$

Then the homomorphisms ρ, $\rho' \colon A \to M_2(A)$, $\rho(x) = e_{11} \otimes x$,[1] $\rho'(x) = e_{22} \otimes x$ are equivariant, and one has:

LEMMA 3. *Let α, u_t and α' be as above; then there exists a unique isomorphism i_u of $A \rtimes_{\alpha'} \mathbb{R}$ with $A \rtimes_{\alpha'} \mathbb{R}$ such that*

$$e_{21} \hat{\rho}(y) e_{12} = \hat{\rho}'(i(y)) \qquad \forall y \in A \rtimes_\alpha \mathbb{R}.$$

Proof. Since ρ is an equivariant isomorphism of A with the reduced algebra $M_2(A)_{e_{11}}$, e_{11} being fixed by β, the dual $\hat{\rho}$ is an isomorphism of $A \rtimes_\alpha \mathbb{R}$ with the reduced algebra of $M_2(A) \rtimes_\beta \mathbb{R}$ by e_{11}. In the same way $\hat{\rho}'$ is an isomorphism of $A \rtimes_{\alpha'} \mathbb{R}$ with $(M_2(A) \rtimes_\beta \mathbb{R})_{e_{22}}$, hence the conclusion.

Q.E.D.

II. CONSTRUCTION OF THE THOM MAP

In this section we introduce three simple axioms that a Thom map should satisfy and prove the existence and uniqueness of the Thom map.

To each C^* dynamical $\mathbb{R}$-system $(A, \mathbb{R}, \alpha)$ we want to associate an additive map ϕ^i_α of $K_i(A)$ in $K_{i+1}(A \rtimes_\alpha \mathbb{R})$, $i \in \mathbb{Z}/2$, in such a way that

[1] $e_{11} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

AXIOM 1. If $A=\mathbb{C}$ (so that $\alpha_t=\mathrm{id}$ $\forall t\in\mathbb{R}$), the image by ϕ_α of the positive generator of $K^0(pt)=\mathbb{Z}$ is the positively oriented generator of $K^1(\hat{\mathbb{R}})=K_1(C^*(\mathbb{R}))=\mathbb{Z}$.

AXIOM 2. (Naturality). If ρ: $A\to B$ is an equivariant homomorphism (not necessarily unital), then

$$(\hat{\rho})_*\circ\phi_\alpha^i=\phi_\beta^i\circ\rho_* \qquad i\in\mathbb{Z}/2.$$

AXIOM 3. (Suspension). Let $S\alpha$ be the suspension of α, then

$$s_{\hat{A}}^{i+1}\circ\phi_\alpha^i=\phi_{S\alpha}^{i+1}\circ s_A^i \qquad i\in\mathbb{Z}/2.$$

Let us first prove consequence of the axioms which will imply the uniqueness of ϕ.

PROPOSITION 1. *Let $(\tilde{A},\mathbb{R},\tilde{\alpha})$ be obtained by adjoining a unit to A, then ϕ_α^i is the restriction of $\phi_{\tilde{\alpha}}^i$ to $K_i(A)=\mathrm{Ker}((\varepsilon_A)_*)$.*

Proof. Let i_A: $A\to\tilde{A}$ be the equivariant inclusion of A in $\tilde{A}$, then $(\hat{i}_A)_*\circ\phi_\alpha^i=\phi_{\tilde{\alpha}}^i\circ(i_A)_*$ by naturality. By Lemma 2 the map $(\hat{i}_A)_*$ is an injection, so this equality determines ϕ_α^i. Q.E.D.

This proposition allows us to restrict our study to the case of unital A. We now compare ϕ_α with ϕ_{α_n}, where $(M_n(A),\mathbb{R},\alpha_n)$ is the C^* dynamical system obtained by tensoring with $M_n(\mathbb{C})$ with trivial action of $\mathbb{R}$. The homomorphism ρ_n: $x\in A\to e_{11}\otimes x$ is equivariant and

PROPOSITION 2. *Let $e\in\mathrm{Proj}\,M_n(A)$, (resp. $u\in GL_n(A)$), then $\phi_\alpha^0([e])$, (resp. $\phi_\alpha^1([u])$), is uniquely determined by the equality*:

$$(\hat{\rho}_n)_*\phi_\alpha^0([e])=\phi_{\alpha_n}^0([e])$$

(resp. $(\hat{\rho}_n)_*\phi_\alpha^1([u])=\phi_{\alpha_n}^1([u])$).

Proof. First $\rho_n(e)\in M_n(M_n(A)_{e_{11}})$ is equivalent to

$$\begin{bmatrix} e & & 0\\ & 0 & \\ & & \ddots \\ 0 & & 0\end{bmatrix}\in M_n(M_n(A)),$$

which represents the class of e in $K_0(M_n(A))$. So $(\rho_n)_*([e]_A)=[e]_{M_n(A)}$. The naturality shows that $\phi_{\alpha_n}^0((\rho_n)_*[e]_A)=(\hat{\rho}_n)_*\phi_\alpha^0([e])$, so it remains to show that $(\hat{\rho}_n)_*$ is an injection. But $\hat{\rho}_n$ is an isomorphism of $A\rtimes_\alpha\mathbb{R}$ with the reduced algebra of $M_n(A) \rtimes_{\alpha_n} \mathbb{R}$ by the multiplier e_{11}, hence giving an isomorphism $(\hat{\rho}_n)_*$ of $K_i(A \rtimes_\alpha \mathbb{R})$ with $K_i(M_n(A) \rtimes_{\alpha_n} \mathbb{R})$. The proof for u instead of e is similar. Q.E.D.

We next show that if we replace the action α of $\mathbb{R}$ on A by an exterior equivalent action α', the map ϕ_α does not change. For this we do not need to assume A unital. So let $(u_t)_{t \in \mathbb{R}}$ be a unitary α-cocycle, $u_t \in M(A)$ $\forall t \in \mathbb{R}$ ([11, p. 356]) and α'_t, $\alpha'_t(x) = u_t \alpha_t(x) u_t^*$ $\forall t \in \mathbb{R}$. Also let i_u be the canonical isomorphism (Lemma 3) of $A \rtimes_\alpha \mathbb{R}$ with $A \rtimes_{\alpha'} \mathbb{R}$.

PROPOSITION 3. $\phi^j_{\alpha'} = (i_u)_* \circ \phi^j_\alpha, j \in \mathbb{Z}/2$.

Proof. Let β be the action of $\mathbb{R}$ on $M_2(A)$ with $\beta_t(e_{11} \otimes x) = e_{11} \otimes \alpha_t(x)$, $\beta_t(e_{21} \otimes 1) = e_{21} \otimes u_t$, $\beta_t(e_{22} \otimes x) = e_{22} \otimes \alpha'_t(x)$ for any $t \in \mathbb{R}$, $x \in A$. Let ρ (resp. ρ') be the equivariant homomorphism $\rho(x) = e_{11} \otimes x$ (resp. $e_{22} \otimes x$) for $x \in A$. By Lemma 3 one has $e_{21}\hat{\rho}(y)e_{12} = \hat{\rho}'(i_u(y))$ $\forall y \in A \rtimes_\alpha \mathbb{R}$, where $e_{ij} \in M_2(\mathbb{C})$ is considered as a multiplier of $M_2(A) \rtimes_\alpha \mathbb{R}$, $i, j = 1, 2$.

Thus, as inner automorphisms act trivially on K, we get $(\hat{\rho})_* = (\hat{\rho}')_* \circ (i_u)_*$, and by naturality

$$\hat{\rho}'_* \circ \phi^j_{\alpha'} = \phi^j_\beta \circ \rho'_* = \phi^j_\beta \circ \rho_* = \hat{\rho}_* \circ \phi^j_\alpha = \hat{\rho}'_* \circ (i_u)_* \circ \phi^j_\alpha,$$

which gives the answer because $\hat{\rho}'_*$ is injective. Q.E.D.

The uniqueness of the Thom map ϕ^0_α (and hence also of ϕ^1_α by Axiom 3) follows from

PROPOSITION 4. *Let $(A, \mathbb{R}, \alpha)$ be a unital C^* dynamical system $e \in \operatorname{Proj} A^\infty$, then there exists an exterior equivalent action α' which fixes e.*

Indeed, then any $e \in \operatorname{Proj} A$ being equivalent (Appendix 3) to an $e' \in \operatorname{Proj} A^\infty$, we can, to compute $\phi^0_\alpha([e])$ assume that e is fixed by α, or in other words that $e = \rho(1)$, where $\rho: \mathbb{C} \to A$ is the equivariant homomorphism $\rho(\lambda) = \lambda e$ $\forall \lambda \in \mathbb{C}$. By Axiom 1, we then get $\phi^0_\alpha([e]) = \hat{\rho}_*([b])$, where $[b]$ is the positively oriented generator of $K_1(C^*(\mathbb{R}))$.

Proof of 4. Let H be the canonical unbounded multiplier of the crossed product $A \rtimes_\alpha \mathbb{R}$ (cf. Appendix), so that for any $x \in A$, one has $\exp(itH)x \exp(-itH) = \alpha_t(x)$ $\forall t \in \mathbb{R}$.

We want to replace H by $H' = eHe + (1-e)\, H(1-e)$, which obviously commutes with e. We have to prove that H' makes sense (the problem being that H is unbounded), and that $\exp(itH')$ normalizes A for all t. The above equality and the definition $\delta(x) = \lim_{t \to 0}((\alpha_t(x) - x)/t)$ show that for $x \in A^\infty$, one has $iHx = ixH + \delta(x)$, and hence $H' = H + P$, $P = i[\delta(e), e]$.

As P is bounded and $P = P^*$, $H' = H + P$ is a selfadjoint unbounded multiplier, by construction H' commutes with e, and generates the one parameter group $\exp(itH') = u_t^P \exp(itH)$, where

$$u_t^P = \sum_0^{\infty} i^n \int_{0 \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant t} \alpha_{s_1}(P) \cdots \alpha_{s_n}(P)\, ds_1 \cdots ds_n.$$

Of course one can, without introducing H or H', define P by the formula $P = i[\delta(e), e]$, u_t^P by the above norm convergent series and check directly that $\alpha'_t = u_t^P \alpha_t(\cdot)(u_t^P)^*$ fixes e, from the equality $\lim_{t \to 0}(1/t)\ (\alpha'_t(e) - e) = \delta(e) + i[P, e] = 0$. Q.E.D.

The above proof shows that eHe is a selfadjoint unbounded multiplier of $A \rtimes_\alpha \mathbb{R}$, for any $e \in \operatorname{Proj} A^\infty$. Thus for any continuous function $b \in \tilde{C}_0(\hat{\mathbb{R}})$, $b(eHe)$ is a multiplier of the reduced algebra $(A \rtimes_\alpha \mathbb{R})_e$. Let us describe it independently of the notion of unbounded multipliers. Recall that $L^1(\mathbb{R}, A)$ is identified with a * subalgebra of the crossed product $A \rtimes_\alpha \mathbb{R}$.

LEMMA 5. *Let* $b = \lambda + \hat{f} \in \tilde{C}_0(\hat{\mathbb{R}})$, *where* $\lambda \in \mathbb{C}$, $f \in L^1(\mathbb{R})$ *then* $b(eHe) - \lambda e \in L^1(\mathbb{R}, A)_e$ *is represented by the map*:

$$t \in \mathbb{R} \to f(-t) e u_t^P \in A.$$

Proof. One has $\hat{f}(eHe) = \hat{f}(eH'e)$, $\hat{f}(H') = \int e^{-itH'} f(t)\, dt = \int f(-t) u_t^P e^{itH}\, dt$, $e\hat{f}(H') = \int f(-t)\, e u_t^P e^{itH}\, dt$. Q.E.D.

If one does not wish to use unbounded multipliers one can of course define $b(eHe)$ by Lemma 5, checking that the formula defines an element of $L^1(\mathbb{R}, A)_e$, for $f \in L^1(\mathbb{R})$, and extends to a homomorphism of $C^*(\mathbb{R}) = C_0(\hat{\mathbb{R}})$ in $(A \rtimes_\alpha \mathbb{R})_e$.

Let us construct ϕ_α^0. Let $e \in \operatorname{Proj} M_n(A^\infty)$, and α_n the action id $\otimes\, \alpha$ of $\mathbb{R}$ on $M_n(A)$, $H_n = 1 \otimes H$ the corresponding unbounded multiplier of $M_n(A) \rtimes_{\alpha_n} \mathbb{R} = M_n(\mathbb{C}) \otimes (A \rtimes_\alpha \mathbb{R})$.

PROPOSITION 6. (a) *The class, in* $K_1(A \rtimes_\alpha \mathbb{R})$ *of the invertible element* $b(eH_n e + Q)$ *of* $GL_1((M_n(A) \rtimes_{\alpha_n} \mathbb{R})_e)^\sim$ *is independent of the choices of* $b = 1 + h \in \tilde{C}_0(\hat{\mathbb{R}})$, *invertible with winding number* 1 *and of* $Q = Q^* \in (M_n(A))_e$.

(b) *Let* $e_1, e_2 \in \operatorname{Proj} M_n(A^\infty)$, $u \in M_n(A^\infty)$ *with* $u^*u = e_1$, $uu^* = e_2$ *then* $u b(e_1 H_n e_1) u^* = b(e_2 H_n e_2 + Q)$, *with* $Q = -iu\delta_n(u^*)e_2$ *for any* $b \in \tilde{C}_0(\hat{\mathbb{R}})$.

(c) *Let* $e_1, e_2 \in \operatorname{Proj} M_n(A^\infty)$, $e_1 e_2 = e_2 e_1 = 0$, *then* $e_1 H e_1 + e_2 H e_2 = (e_1 + e_2) H(e_1 + e_2) + Q$, *where* $Q = -i(e_1 \delta_n(e_2) + e_2 \delta_n(e_1)) \in M_n(A)_{(e_1 + e_2)}$.

Proof. (a) For each $\lambda \in [0, 1]$, $b(eH_n e + \lambda Q)$ is invertible in $(M_n(A) \rtimes_{\alpha_n} \mathbb{R})_e^\sim$, so we get a norm continuous (use Lemma 5) arc in GL_1 joining $b(eH_n e)$ and $b(eH_n e + Q)$. The independence of b is clear.

(b) One has $ub(e_1 H_n e_1)u^* = b(uH_n u^*)$ and also

$$uH_n u^* = e_2 H_n e_2 - ie_2(u\delta_n(u^*)e_2.$$

(c) We have $e_1 H_n e_2 = e_1(H_n e_2 - e_2 H_n) = -ie_1 \delta_n(e_2)$. Q.E.D.

If in Proposition 6a we replace n by $n' > n$, e by $\left[\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right] \in M_{n'}(A^\infty)$, we do not change the reduced algebra $(M_n(A) \rtimes_{\alpha_n} \mathbb{R})_e$, nor the element $b(eHe)$ of $(M_n(A) \rtimes_{\alpha_n} \mathbb{R})_e^{\sim}$. If we replace e by a projection $e' \in M_n(A^\infty)$, equivalent to e in $M_n(A)$, then e' is equivalent to e in $M_n(A^\infty)$ (Appendix 3), and Proposition 6b shows that we do not change the element of $K_1(A \rtimes_\alpha \mathbb{R})$ defined by $b(eHe)$: we have $[b(eHe)] = [b(e'He')]$. Finally, Proposition 6c shows that the class of $b((e_1 + e_2)\, H(e_1 + e_2))$ for $e_1 e_2 = e_2 e_1 = 0$ is the same as the sum in $K_1(A \rtimes_\alpha \mathbb{R})$ of the classes of $b(e_i He_i)$, since obviously $[b(e_1 He_1 + e_2 He_2)] = [b(e_1 He_1)] + [b(e_2 He_2)]$. Thus there exists a well-defined additive map ϕ_α^0 of $K_0(A)$ in $K_1(A \rtimes_\alpha \mathbb{R})$.

If $A = \mathbb{C}$, with $\alpha_t = \mathrm{id}$ $\forall t \in \mathbb{R}$, we have $A \rtimes_\alpha \mathbb{R} = C^*(\mathbb{R}) = C_0(\hat{\mathbb{R}})$ and by construction $\phi_\alpha^0([1])$, where $1 \in \operatorname{Proj} A$ is given by the class of $b(H)$ with b as in 6a, i.e., is the positively oriented generator of $K_1(C_0(\hat{\mathbb{R}}))$.

If ρ: $A \to B$ is an equivariant homomorphism, and $e \in \operatorname{Proj} M_n(A^\infty)$, then $\rho_n(e) \in \operatorname{Proj}\ M_n(B^\infty)$, where $\rho_n = \mathrm{id} \otimes \rho$. Also $(\hat{\rho})_n = \mathrm{id} \otimes \hat{\rho}$ is a homomorphism of $M_n(A) \rtimes_{\alpha_n} \mathbb{R}$ in $M_n(B) \rtimes_{\beta_n} \mathbb{R}$ and one has

$$\hat{\rho}_n(b(eH_n^A e) = b(\rho_n(e)\, H_n^B \rho_n(e))$$

and hence

$$\hat{\rho}_* \phi_\alpha^0(e) = \phi_\beta^0(\rho_*[e]).$$

This naturality of ϕ^0 allows us using Lemma 2, to define it also when A is not unital, by $\phi_\alpha^0 = (\hat{\imath}_A)_*^{-1} \phi_{\tilde{\alpha}}^0$. Then the naturality remains true, for not necessarily unital homomorphisms ρ: $A \to B$, since $\tilde{\rho}$: $\tilde{A} \to \tilde{B}$ satisfies $\tilde{\rho} \circ i_A = i_B \circ \rho$, $\tilde{\rho} \circ \hat{\imath}_A = \hat{\imath}_B \circ \hat{\rho}$ and $\phi_{\tilde{\beta}}^0 \circ \tilde{\rho} = (\hat{\tilde{\rho}}) \circ \phi_{\tilde{\alpha}}^0$.

Now Axiom 3 defines ϕ_α^1 uniquely by the equality

$$s_{\hat{A}}^0 \phi_\alpha^1([u]) = \phi_{S\alpha}^0(s_A^1([u])).$$

If ρ: $A \to B$ is a homomorphism, then $S\rho = \mathrm{Id} \otimes \rho$ is a homomorphism of SA in SB and the equalities $(S\rho)_* \circ s_A^i = s_B^i \circ \rho_*$, $i = 1, 2$ (which follows from Appendix 1) and $(S\rho)\hat{\ } = S\hat{\rho}$, show that

$$\begin{aligned}\phi_\beta^1(\rho_*(u)) &= (s^0)^{-1} \phi_{S\beta}^0(s^1 \rho_*(u)) = (s^0)^{-1} \phi_{S\beta}^0((S\rho)_* s^1(u)) \\ &= (s^0)^{-1} (S\hat{\rho})_* \phi_{S\alpha}^0(s^1(u)) = \hat{\rho}_* \phi_\alpha^1(u) \qquad \forall u \in K_1(A).\end{aligned}$$

This shows that ϕ satisfies Axioms 1 and 2, and also 3 for $i=1$, it remains to check 3 for $i=0$, i.e., to show that

$$s^1 \circ \phi^0 = \phi^1_{S\alpha} \circ s^0.$$

By construction one has $\phi^1_{S\alpha} = (s^0)^{-1} \phi^0_{S^2\alpha} \circ s^1$, so one has to show that $s^0 \circ s^1 \circ \phi^0_\alpha = \phi^0_{S^2\alpha} \circ s^1_0 \circ s^0$. Now $s^i \circ s^{i+1}$ coincides with the external tensor product by the Bott element $\beta \in K_0(C_0(\mathbb{R}^2))$ (Appendix 2), so the next proposition ends the proof of existence for ϕ.

PROPOSITION 7. *Let C be a C^* algebra, and $\beta = \mathrm{id} \otimes \alpha$ the action of $\mathbb{R}$ on $C \otimes_{\min} A$, and identify $(C \otimes_{\min} A) \rtimes_\beta \mathbb{R}$ with $C \otimes_{\min} (A \rtimes_\alpha \mathbb{R})$, then for $c \in K_i(\mathbb{C})$, $a \in K_j(A)$, $i, j \in \mathbb{Z}/2$ one has*

$$\phi_\beta(c \boxtimes a) = c \boxtimes \phi_\alpha(a).$$

Proof. As we need it only for $i=j=0$ we shall spell out the proof only in this case. Considering the homomorphism $i_C \otimes \mathrm{id}$ of $C \otimes A$ in $\tilde{C} \otimes A$ and using the naturality of ϕ and the equivariantly split exact sequence: $0 \to C \otimes A \to \tilde{C} \otimes A \to A \to 0$, one can replace C by $\tilde{C}$, i.e., assume that C is unital. In the same way one can assume that A is unital.

Let then $f \in \operatorname{Proj} M_p(C)$, $e \in \operatorname{Proj} M_q(A)$, with the obvious notations $b(f \otimes e)\,(1 \otimes H_q)\,(f \otimes e)) = b(f \otimes eH_q e) = f \otimes b(eH_q e)$. Q.E.D.

Remark. Intrinsic description of ϕ^0_α.

In this remark we shall point out the analogy between the construction of ϕ^0_α and the classical constructions of characteristic classes of vector bundles involving an auxiliary choice of connection. To do this, we describe $K_0(A)$, where A is a unital C^* algebra, as a group with one generator $[\mathscr{E}]$ for each isomorphism class of finite[2] projective module $\mathscr{E}$ and one relation $[\mathscr{E}_1] + [\mathscr{E}_2] = [\mathscr{E}_1 + \mathscr{E}_2]$ for each pair of such modules. Given an idempotent $e \in M_n(A)$ its image eA^n is a right A-module (finite and projective) and if e is selfadjoint, $e = e^*$, this module has a natural hermitian structure, i.e., cf. [5, p. 600], a strictly positive hermitian form $\langle \xi, \eta \rangle \in A$, $\forall \xi, \eta \in \mathscr{E}$, which in this case is given by

$$\langle \xi, \eta \rangle = \sum \eta_i^* \xi_i \in A, \qquad \forall \xi, \eta \in eA^n.$$

Next if $(A, \mathbb{R}, \alpha)$ is a C^* dynamical system, and $A^\infty = \{x \in A,\ t \to \alpha_t(x)$ is of class $C^\infty\}$, Appendix 3 shows that any finite projective module $\mathscr{E}$ on A is isomorphic to a smooth one, i.e., the image (by the inclusion of A^∞ in A) of

[2] We say finite instead of finitely generated; the modules will be always right modules.

a finite projective module $\mathscr{E}^\infty$ on A^∞. A smooth hermitian structure on $\mathscr{E}^\infty$ is then simply a strictly positive hermitian form $\langle\xi,\eta\rangle\in A^\infty$, $\forall\xi,\ \eta\in\mathscr{E}^\infty$. To such an object we want to associate an element of $K_1(A\rtimes_\alpha\mathbb{R})$. For this we need also an intrinsic description of K_1. Since $J=A\rtimes_\alpha\mathbb{R}$ is not unital, the description uses the excision property in K theory: given a surjective homomorphism ρ: $B\to B'$ of unital C^* algebras, with kernel J, the relative group $K_1(B,B')$ is equal to $K_1(\tilde{J})$. By definition, $K_1(B,B')$ has one generator $[T]$ for each isotopy class of automorphisms $T\in\operatorname{Aut}_B(\mathscr{E})$ of a finite projective module $\mathscr{E}$ over B, such that $\rho(T)=\mathrm{id}$. It has one relation $[T_1]+[T_2]=[T_1+T_2]$ for each pair, and one $[\mathrm{id}_{\mathscr{E}}]=0$ for each finite projective module $\mathscr{E}$ over B. The isomorphism $K_1(B,B')\to K_1(J)$ associates to $T\in\operatorname{Aut}_B(\mathscr{E})$ the connected component in $GL(\tilde{J})$ of the element $W(T+\mathrm{id}_{\mathscr{E}'})W^{-1}$ of $GL_n(J)$, where $\mathscr{E}+\mathscr{E}'$ is free and W is an isomorphism of $\mathscr{E}+\mathscr{E}'$ with B^n. We let $B=A\,\tilde{\rtimes}_\alpha\,\mathbb{R}$ be the C^* subalgebra of $M(A\rtimes_\alpha\mathbb{R})$ generated by $A\rtimes_\alpha\mathbb{R}$ and A, so we have a split exact sequence:

$$0\to A\rtimes_\alpha\mathbb{R}\to A\,\tilde{\rtimes}_\alpha\,\mathbb{R}\xrightarrow{\rho}A\to 0.$$

The retraction r is the usual inclusion of A in $M(A\rtimes_\alpha\mathbb{R})$. Now let $\mathscr{E}$ be a finite projective module on A, and $r_*(\mathscr{E})=\mathscr{E}\otimes_A B$ the corresponding module over $B=A\,\tilde{\rtimes}_\alpha\,\mathbb{R}$. Let $\mathscr{E}\rtimes_\alpha\mathbb{R}$ be the linear span in $r_*(\mathscr{E})$ of the $\xi\otimes_A b$, $\xi\in\mathscr{E}$, $b\in A\rtimes_\alpha\mathbb{R}=J$, as $\rho(r(a)b)=a\rho(b)$ $\forall a\in A$, $b\in B$, $\mathrm{id}\otimes_A\rho$ makes sense and one has the split exact sequence

$$0\longrightarrow\mathscr{E}\rtimes_\alpha\mathbb{R}\longrightarrow r_*(\mathscr{E})\xrightarrow{\mathrm{id}\otimes\rho}\mathscr{E}\longrightarrow 0.$$

Let $L^1(\mathbb{R},\mathscr{E})$ be the right $L^1(\mathbb{R},A)$ module defined by

$$(\xi*a)(t)=\int\xi(s)\,\alpha_s(a(t-s))\,ds\qquad\forall\xi\in L^1(\mathbb{R},\mathscr{E}),\,a\in L^1(\mathbb{R},A).$$

As $L^1(\mathbb{R},\mathscr{E})=\mathscr{E}\otimes_A L^1(\mathbb{R},A)$, one passes from $L^1(\mathbb{R},\mathscr{E})$ to $\mathscr{E}\rtimes_\alpha\mathbb{R}$ exactly as from $L^1(\mathbb{R},A)$ to $A\rtimes_\alpha\mathbb{R}$.

Now let $\mathscr{E}$ be smooth and hermitian, ∇ be a compatible connection (cf. [5, Definition 2]), then for any $f\in L^1(\mathbb{R})$ the equality $(f*\xi)(t)=\int f(s)\,\theta_s(\xi(t-s))\,ds$ $\forall\xi\in L^1(\mathbb{R},\mathscr{E})$, where $\theta_s=\exp s\nabla$ is the parallel transport associated to ∇, defines uniquely an endomorphism $\hat{f}(i\nabla)$ of $\mathscr{E}\rtimes_\alpha\mathbb{R}$. The equality $f*(\xi*a)=(f*\xi)*a$ $\forall a\in A\rtimes_\alpha\mathbb{R}$ follows from $\theta_s(\xi a)=\theta_s(\xi)\,\alpha_s(a)$ $\forall\xi\in\mathscr{E}$, $a\in A$, $s\in\mathbb{R}$.

The content of this remark is that for any $f\in L^1(\mathbb{R})$, such that $b=1+\hat{f}$ is invertible with winding number 1, one has

$$\phi_\alpha^0[\mathscr{E}]=[b(i\nabla)]\in K_1(A\rtimes_\alpha\mathbb{R}).$$

Then Proposition 6a means that the choice of connection ∇ does not affect $[b(i\nabla)]$, and the choice made there, $b(eHe)$, corresponds to the Grassmannian connection $\nabla\xi = e\delta_n(\xi)$ on the finite projective module $\mathscr{E} = eA^n$.

III. Computation of $\hat{\tau} \circ \phi_\alpha^1$

In this section we shall first give an explicit formula for $\phi_\alpha^1([u])$, $u \in GL(A^\infty)$, as a loop in $GL(A \tilde{\rtimes}_\alpha \mathbb{R})$, and then use it to compute, given an α-invariant trace τ on A, the composition of ϕ_α^1 with the dual trace $\hat{\tau}$. (Recall that any reasonable trace defines an additive map of K_0 to $\mathbb{R}$.)

By definition $\phi_\alpha^1 = (s^0)^{-1}\phi_{S\alpha}^0 s^1$; to compute it, we can, using Proposition 2 of Section II, assume that $u \in GL_1(A^\infty)$ (instead of GL_n), also, in order to handle only selfadjoint idempotents we take u to be unitary, assuming first that A is unital.

We let $SA = C_0(I) \otimes A$, where $I =]0, \pi[$, and we identify the C^* algebra $(SA)^\sim$ with the subalgebra of $C(\bar{I}) \otimes A$:

$$(SA)^\sim = \{x \in C(\bar{I}) \otimes A, x(0) = x(\pi) \in \mathbb{C} \subset A\}.$$

The action $\tilde{S}\alpha$ of $\mathbb{R}$ is then the restriction of the action $\beta = \mathrm{id} \otimes \alpha$,

$$(\beta_t(x))(s) = \alpha_t(x(s)) \qquad \forall x \in C(\bar{I}) \otimes A,\ \forall t \in \mathbb{R},\ s \in \bar{I}.$$

By Appendix 1, $s^1([u]) = [e] - [e_0]$, where e_0 (resp. e) are projections belonging to $M_2(\widetilde{SA})$, given by

$$e_0(s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \forall s \in \bar{I}, \qquad e(s) = W(s)e_0(s)W(s)^*,$$

where $\forall s \in [0, \pi/2]$, $W(s) = \begin{bmatrix} u^* & 0 \\ 0 & 1 \end{bmatrix}$

$$\text{and} \qquad \forall s \in \left[\frac{\pi}{2}, \pi\right]\ W(s) = R(s)\begin{bmatrix} u^* & 0 \\ 0 & 1 \end{bmatrix} R(s)^*,$$

$$R(s) = \begin{bmatrix} \sin s & \cos s \\ -\cos s & \sin s \end{bmatrix}.$$

We want to compute $\phi_{\tilde{S}\alpha}^0([e])$. We identify the crossed product B of $\widetilde{SA}$ by the action of $\mathbb{R}$ with a subalgebra of $(C(\bar{I}) \otimes A) \rtimes_\beta \mathbb{R} = C(\bar{I}) \otimes (A \rtimes_\alpha \mathbb{R})$, so that any element of, say, $M_2(B)$, is viewed as a continuous map from $\bar{I}$ to the algebra $M_2(A \rtimes_\alpha \mathbb{R})$, and in a similar way any element of $M_2(\tilde{B})$ is a continuous map from $\bar{I}$ to $M_2((A \rtimes_\alpha \mathbb{R})^\sim)$. In particular, letting H be the canonical unbounded multiplier of the crossed product $A \rtimes_\alpha \mathbb{R}$, one checks that the canonical unbounded multiplier H' of $B = SA \rtimes_\alpha \mathbb{R}$ is given by the constant map $s \in \bar{I} \to H$.

Fixing the continuous function $b \in \tilde{C}_0(\hat{\mathbb{R}})$, invertible with winding number 1, as in Proposition 6a, we now have to compute $b(eH'_2 e)$. By Proposition 6a we can replace it by $b(e(H'_2 + Q)e)$ provided that $Q = Q^*$ and $eQe \in M_2(\widetilde{SA})$. We choose Q such that, for $s \in [\pi/2, \pi]$, one has

$$H_2 + Q(s) = W(s)\, H_2\, W(s)^*.$$

So, for $s \in [\pi/2, \pi]$, one gets $Q(s) = i\delta_2(W^*(s))\, W(s) = R(s)\ \left[\begin{smallmatrix} P & 0 \\ 0 & 0 \end{smallmatrix}\right]\ R(s)^*$ where $P = i\delta(u^*)u$ (and $\delta(u^*) = \lim_{t\to 0}(1/t)\ (\alpha_t(u^*) - u^*)$ exists since by hypothesis $u \in A^\infty$).

We define $Q(s)$ for $s \in [0, \pi/2]$, in such a way that eQe belongs to $M_2(\widetilde{SA})$. As $e(\pi)\, Q(\pi)\, e(\pi) = 0$ is already chosen, we need to have $Q(0) = 0$ so we choose

$$Q(s) = \frac{2s}{\pi}\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \qquad \forall s \in \left[0, \frac{\pi}{2}\right].$$

We are now prepared to prove:

PROPOSITION 1. *$\phi^1_\alpha([u])$ is the element of $K_0(A \rtimes_\alpha \mathbb{R})$ represented by the composition of the following two paths in $GL_2(A \tilde{\rtimes}_\alpha \mathbb{R})$:*

$$\mathscr{C}_1(\lambda) = \begin{bmatrix} b(H + \lambda P)b(H)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \qquad \lambda \in [0, 1],$$

$$\mathscr{C}_2(t) = W(t)\begin{bmatrix} b(H) & 0 \\ 0 & 1 \end{bmatrix} W(t)^{-1}\begin{bmatrix} b(H)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \qquad t \in \left[\frac{\pi}{2}, \pi\right].$$

Proof. First $b(e_0 H'_2 e_0)$ corresponds to the constant map $s \to b(H)$ of $\bar{I}$ in $GL_1(A \tilde{\rtimes}_\alpha \mathbb{R})$. We then have to compute $(1 - e) + b(e(H'_2 + Q)e)$ as a path in $GL_2(A \tilde{\rtimes}_\alpha \mathbb{R})$ and to multiply it by

$$\begin{bmatrix} b(H)^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

For $s \in [0, \pi/2]$, we have $e(s) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $Q(s) = \left[\begin{smallmatrix} \lambda P & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\lambda = 2s/\pi$ so we get the matrix $\left[\begin{smallmatrix} b(H+\lambda P) & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

For $s \in [\pi/2, \pi]$ we have $e(s) = W(s)\ \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\ W(s)^*$, $H'_2 + Q(s) = W(s)\ H'_2\, W(s)^*$ so we get the matrix $W(s)\ \left[\begin{smallmatrix} b(H) & 0 \\ 0 & 1 \end{smallmatrix}\right]\ W(s)^{-1}$. Q.E.D.

Remark 2. Assume now that A is not unital, let $u \in \tilde{A}^{\infty}$ be unitary, then the above path represents, by Proposition 1, the image of $\phi^1_\alpha([u])$ in $K_1(\tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{R})$. We claim that the path belongs in fact to $GL_2(A \tilde{\rtimes}_\alpha \mathbb{R})$, and hence represents, as it stands, $\phi^1_\alpha([u])$. To see this, as the construction of the path is natural, it is enough to show that when $A = \mathbb{C}$, $u = 1$, it gives the constant path 1. But if $u = 1$ one has $P = 0$, $W(s) = 1$, hence the conclusion.

THEOREM 3. *Let τ be a trace*[3] *on A, fixed by α_t, $t \in \mathbb{R}$, u a unitary of $\tilde{A}$ of the form $u = 1 + z$, with $z \in A^{\infty}$, z and $\delta(z)$ in the domain of τ. Let $\hat{\tau}$ be the dual trace on $A \times_\alpha \mathbb{R}$ then*

$$\hat{\tau}(\phi^1_\alpha([u])) = \frac{1}{2i\pi}\,\tau(\delta(u)u^*).$$

Proof. For the proof we shall need a more careful choice of $b \in C_0^{\sim}(\hat{\mathbb{R}})$ than above. Let l be a rational function of $z \in \mathbb{C}$, with a zero of order 2 at $+\infty$, such that $b = 1 + l$ has no real pole or zero and has winding number equal to 1. Take for instance

$$l(z) = \left(\frac{z-i}{z+i} - 1\right)\frac{1}{1 - i\varepsilon z} \qquad \text{for } \varepsilon \text{ small enough.}$$

Let $\mathscr{C}_1$, $\mathscr{C}_2$ be the paths of Proposition 1. We want to show that they satisfy the hypothesis of Appendix 4, and compute the two integrals

$$I_j = \frac{1}{2i\pi}\int \hat{\tau}(\mathscr{C}_j'(t)\,\mathscr{C}_j(t)^{-1}).$$

We shall get that $I_1 = (1/2i\pi)\,\tau(\delta(u)u^*)$, and $I_2 = 0$.

Proof that $I_2 = 0$. We have $\mathscr{C}_2(t) = W(t)(1+Z)\;W(t)^{-1}(1+Z)^{-1}$, where

$$Z = \begin{bmatrix} l(H) & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad W(t) = R(t)\begin{bmatrix} u^{-1} & 0 \\ 0 & 1 \end{bmatrix} R(t)^*.$$

As u is equal to 1 modulo the ideal $\operatorname{Dom}\tau$ of definition of τ, we get that $W'(t) \in M_2(\operatorname{Dom}\tau)$. As $l \in L^1(\mathbb{R})$, it follows that $W'(t)Z \in M_2(\operatorname{Dom}\hat{\tau}) = \operatorname{Dom}\hat{\tau}_2$, the same holds for $W'(t)Z'$, where $1 + Z' = (1+Z)^{-1}$. We have $\mathscr{C}_2(t)'\;\mathscr{C}_2(t)^{-1} = W'(t)\;\;W(t)^* + W(t)\;\;(1+Z)\;\;W'(t)^*\;\;W(t)\;\;(1+Z)^{-1}$ $W(t)^* = W(t)\;(W(t)^*\;W'(t) - (1+Z)\;W(t)^*\;W'(t))\;(1+Z)^{-1})\;W(t)^*$. So $\mathscr{C}_2(t)' \in \operatorname{Dom}\hat{\tau}_2$ and $\hat{\tau}_2(\mathscr{C}_2(t)'\;\mathscr{C}_2\,(t)^{-1}) = 0$. Q.E.D.

[3] By this we shall mean a semi-continuous semi-finite trace on A; we do not assume that A is unital.

Proof that $I_1 = (1/2i\pi)\,\tau(\delta(u)u^*)$. We have $\mathscr{C}_1(\lambda) = b(H+\lambda P)\ \ b(H)^{-1}$. By hypothesis, b is a rational function of the form $b(z) = 1 + \sum_1^n\ \lambda_i/(z-p_i)$, where we assumed for convenience that all its poles are simple. We want to show that $\hat{\tau}(d/d\lambda(b(H+\lambda P))\ \ b(H+\lambda P)^{-1}) = i\tau(P)$, $\forall \lambda \in [0, 1]$. First we may do it only if $\lambda = 0$. Indeed replace the action α of $\mathbb{R}$ by the exterior equivalent action $\alpha_t^{\lambda P}(x) = u_t^{\lambda P}\alpha_t(x)\quad (u_t^{\lambda P})^*$, where $u_t^{\lambda P} = \sum i^n \int \alpha_{s_1}(P) \cdots \alpha_{s_n}(P)\ ds_1 \cdots ds_n$, $s_1 \leqslant \cdots \leqslant s_n \leqslant t$, then H is changed in $H + \lambda P$ (up to the canonical isomorphism i of Lemma 3) and the dual trace $\hat{\tau}$ is unchanged. Next, we have

$$\left(\frac{d}{d\lambda}(b(H+\lambda P))_{\lambda=0} = \sum_{i=1}^{n} \lambda_i (H-p_i)^{-1}\ P(H-p_i)^{-1}\right).$$

We first have to show that it belongs to the domain of $\hat{\tau}$, in fact, each $(H-p_i)^{-1}\ P(H-p_i)^{-1}$ belongs to this domain. Let indeed $Q \in A$ be such that $\tau(Q^*Q) < \infty$, the element $Q(H-p_i)^{-1} = X$ of $L^2(\mathbb{R}, A)$ given by $X(s) = f(-s)Q$ (where $f \in L^2(\mathbb{R})$ has Fourier transform $\hat{f}$, $\hat{f}(z) = (z-p_i)^{-1}$), satifies $\hat{\tau}(X^*X) = \int |f(s)|^2\ \tau(Q^*Q)\, ds < \infty$. Using the tracial property of $\hat{\tau}$ and the equality:

$$b'(z) = -\sum_{i=1}^{n} \lambda_i (z-p_i)^{-2}$$

we get

$$\hat{\tau}\left(\left(\frac{d}{d\lambda} b(H+\lambda P)\right)_{\lambda=0} b(H+\lambda P)^{-1}\right) = \hat{\tau}(Pb'(H)\, b(H)^{-1}).$$

By our choice of b, b' belongs to $L^1(\hat{\mathbb{R}})$, and the right term of the equality is given by $\tau(P)\, f(0)$, where $\hat{f} = b'b^{-1}$. So $f(0) = (1/2\pi)\ \int\ b'(t)\ b(t)^{-1}\ dt = i$ (winding number of $b) = i$, and we have shown that $I_1 = (1/2i\pi)$ $(i\tau(P)) = (1/2i\pi)\ \tau(\delta(u)u^*)$. Q.E.D.

IV. The Thom Isomorphism

In this section we shall use the duality of Takesaki–Takai for crossed products of C^* algebras by abelian groups, to show that the Thom map constructed in Section II is actually an isomorphism of $K_i(A)$ with $K_{i+1}(A \rtimes_\alpha \mathbb{R})$. This will be done, essentially, by checking that $\phi_{\hat{\alpha}} \circ \phi_\alpha = \mathrm{id}$ where $\hat{\alpha}$ is the dual action.

So, let $(A, \mathbb{R}, \alpha)$ be a C^* dynamical $\mathbb{R}$-system, recall that the dual action, $\hat{\alpha}$, fixes A pointwise and satisfies $\hat{\alpha}_s(U_t) = e^{ist}U_t$ where $U_t = \exp it\, H$ is the canonical representation of $\mathbb{R}$ in $M(A \rtimes_\alpha \mathbb{R})$. It follows that if α' is exterior equivalent to α, and i: $A \rtimes_\alpha \mathbb{R} \to A \rtimes_{\alpha'} \mathbb{R}$ is the canonical isomorphism of Lemma 3, one has $\hat{\alpha}'_s \circ i = i \circ \hat{\alpha}_s$; in other words, i is equivariant. Let now t_α be the canonical isomorphism of the double crossed product $(A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}$ with $A \otimes k$, where k is the C^* algebra of compact operators in $L^2(\mathbb{R})$.

LEMMA 1. *Given α' exterior equivalent to α, there exists a unitary multiplier W of $A \otimes k$ such that, with $\theta = \operatorname{int} W \in \operatorname{Aut}(A \otimes k)$, one has $\theta \circ t_{\alpha'} \circ \hat{\imath} = t_\alpha$.*

Proof. Let β be the action of $\mathbb{R}$ on $M_2(A)$ as in Lemma 3. Let ρ (resp. ρ') be the canonical isomorphism of $A \rtimes_\alpha \mathbb{R} \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}$ (resp. $A \rtimes_{\alpha'} \mathbb{R} \rtimes_{\hat{\alpha}'} \hat{\mathbb{R}}$) on the reduced algebra of $M_2(A) \rtimes_\beta \mathbb{R} \rtimes_{\hat{\beta}} \hat{\mathbb{R}}$ by e_{11} (resp. e_{22}). For the three double crossed products under considerations consider the canonical representations U, V of $\mathbb{R}$, $\hat{\mathbb{R}}$ in the multipliers, and to distinguish them put U, V for α, U', V' for α', U'', V'' for β. Then we have $\rho(U_t) = U''_t e_{11} = e_{11} U''_t$, $\rho(V_s) = V''_s e_{11} = e_{11} V''_s$ and similarly for ρ'.

For $a \in A$ one has $\rho(a) = ae_{11}$, $\rho'(a) = ae_{22}$, and hence $e_{21}\rho(x)e_{12} = \rho'(\hat{\imath}(x))$ $\forall x \in \hat{A} \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}$, since $e_{21} U''_t e_{12} = e_{21}\beta_t(e_{12})U''_t = e_{22}u^*_t U''_t = \rho'(\hat{\imath}(U_t))$ and $e_{21} V''_s\, e_{12} = e_{22} V''_s = \rho'(\hat{\imath}(V_s))$. Let t_β be the canonical isomorphism of $M_2(A) \rtimes_\beta \mathbb{R} \rtimes_{\hat{\beta}} \hat{\mathbb{R}}$ with $M_2(A) \otimes k$. As both e_{11}, e_{22} are fixed by β one has $t_\beta(e_{jj}) = e_{jj} \otimes 1$, $j = 1, 2$, so that $t_\beta(e_{21}) = \left[\begin{smallmatrix} 0 & 0 \\ W & 0 \end{smallmatrix}\right]$ for a suitable unitary multiplier W of $A \otimes k$. By construction of t_α, $t_{\alpha'}$, t_β we have

$$t_\beta(\rho(x)) = \begin{bmatrix} t_\alpha(x) & 0 \\ 0 & 0 \end{bmatrix}, \qquad \forall x \in A \rtimes_\alpha \mathbb{R} \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}$$

and similarly for ρ'. Thus, for x as above one has

$$\begin{bmatrix} 0 & 0 \\ 0 & t_{\alpha'}(\hat{\imath}(x)) \end{bmatrix} = t_\beta(\rho'(\hat{\imath}(x))) = t_\beta(e_{21}\rho(x)e_{12}) = \begin{bmatrix} 0 & 0 \\ W & 0 \end{bmatrix} \begin{bmatrix} t_\alpha(x) & 0 \\ 0 & 0 \end{bmatrix}$$
$$\times \begin{bmatrix} 0 & W^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Wt_\alpha(x)W^* \end{bmatrix}$$

and hence $t_{\alpha'}(\hat{\imath}(x)) = Wt_\alpha(x)W^*$. Q.E.D.

THEOREM 2. *Let $(A, \mathbb{R}, \alpha)$ be a C^* dynamical system. Identify $K(A \otimes k)$ with $K(A)$ by the inverse of the external tensor product by the positive generator of $K_0(k) = \mathbb{Z}$.*

(a) *The map $(t_\alpha)_* \circ \phi^{i+1}_{\hat{\alpha}} \circ \phi^i_\alpha$ is identity from $K_i(A)$ to itself.*

(b) *ϕ^i is an isomorphism of $K_i(A)$ with $K_{i+1}(A \rtimes_\alpha \mathbb{R})$.*

Proof. (a) Put $\Psi^i = (t_\alpha)_* \circ \phi_{\hat\alpha}^{i+1} \circ \phi_\alpha^i$. For any homomorphism $\rho: A \to B$ one checks from the construction of t_α that $t_\alpha \circ \hat{\hat\rho} = (\rho \otimes \mathrm{id}) \circ t_\alpha$, using the naturality of ϕ this shows that Ψ_α^i is natural, with the above identification of $K(A \otimes k)$ with $K(A)$ one gets $\Psi_\beta^i \circ \rho_* = \rho_* \circ \Psi_\alpha^i$. Next if α' is exterior equivalent to α, Lemma 1 shows that $(t_{\alpha'})_* \circ \hat{\imath}_* = (t_\alpha)_*$ and Proposition 2 of Section II shows then that $\hat{\imath}_* \circ \phi_{\hat\alpha} \circ \phi_\alpha = \phi_{\hat\alpha'} \circ \phi_{\alpha'}$ (using the equivariance of i for the dual actions) and hence $\Psi_\alpha^i = \Psi_{\alpha'}^i$, $i \in \mathbb{Z}/2$. To show that $\Psi_\alpha^0([e]) = [e]$, for $[e] \in K_0(A)$, one can, as in the proof of uniqueness in Section II take A unital, and $e \in \operatorname{Proj} A$. Then by Proposition II, one can replace α by α' fixing e, i.e., as $\Psi_\alpha = \Psi_{\alpha'}$ assume that $e = \rho(1)$ where ρ is an equivariant homorphism of $\mathbb{C}$ in A. To see finally that $\Psi_{\mathbb{C}} = \mathrm{id}$ we have to compute $\Psi_\alpha^0([1])$. Let τ be the trace on $C_0(\hat{\mathbb{R}}) = C^*(\mathbb{R}) = \mathbb{C} \rtimes \mathbb{R}$ given by the Plancherel measure on $\mathbb{R}$, then $\hat\tau$ is the usual trace on $k = C_0(\hat{\mathbb{R}}) \rtimes_{\hat{\mathrm{id}}} \hat{\mathbb{R}}$ the double crossed product, in particular it maps $K_0(k)$ injectively to $\mathbb{R}$. Thus, since by Axiom 1, $\phi_\alpha^0([1])$ is given by the canonical generator of $K_1(C_0(\hat{\mathbb{R}}))$, one sees from the formula:

$$\frac{1}{2i\pi}\,\tau(\delta(u)u^*) = \frac{1}{2i\pi}\int_{-\infty}^{\infty} u'(t)\,u(t)^{-1}\,dt = 1,$$

where δ is the generator of the dual action of $\hat{\mathbb{R}}$ on $C_0(\hat{\mathbb{R}})$ (One has $(\hat{\mathrm{id}}_s(f))(t) = f(t+s)$ $\forall t$, $s \in \hat{\mathbb{R}}$), that by Theorem 3 of Section III, $\hat\tau(\phi_{\hat{\mathrm{id}}}^1([u])) = 1$, i.e., $\Psi_\alpha^0([1]) = [1]$. This proves (a) for Ψ_α^0. Using Axiom 3 one gets that $s^1 \circ \Psi_\alpha^1 = \Psi_{S\alpha}^0 \circ s^1$ thus $\Psi_\alpha^1 = \mathrm{id}$.

(b) By (a) it follows that ϕ_α^i is injective, so also $\phi_{\hat\alpha}^{i+1}$ and hence ϕ_α^i is also surjective. Q.E.D.

V. Applications

In this section we apply the results of the last ones, mainly Theorem 3 of Section III and Theorem 2 of Section IV to show that for flows the range of the trace of the crossed product is the same as the range of the topological index of [4]. Since we did not develop in this paper the pseudodifferential calculus for crossed products we shall check the above equality simply by comparing the two formulas.

Let us first state an easy corollary of Theorem 3 of Section III and Theorem 2 of Section IV:

COROLLARY 1. *Let $(A, \mathbb{R}, \alpha)$ be a C^* dynamical system with an α-*

invariant[4] *trace* τ. *Then the image of* $K_0(A \rtimes_\alpha \mathbb{R})$ *by the dual trace* $\hat{\tau}$ *is the group of real numbers of the form*:

$$\frac{1}{2i\pi}\tau_n(\delta_n(u)u^{-1}),$$

where $u \in GL_n(\tilde{A})$, $u = 1 + h$, $\delta_n(h) \in \operatorname{Dom} \tau_n$.

Proof. Combine Theorem 3 of Section III with Theorem 2 of Section IV. Q.E.D.

Let now V be a compact manifold of dimension n, X a smooth vector field on V and α_t the corresponding one parameter group of automorphisms of $C(V)$. Each invariant probability measure μ on V defines a one-dimensional current C on V by the equality $C(\omega) = \int \omega(X)\, d\mu$, where $\omega \in \Omega^1(V)$ is a smooth 1-form. This Ruelle–Sullivan current is closed, and its homology class $[C] \in H_1(V, \mathbb{R})$ is the asymptotic cycle of μ.

COROLLARY 2. *The range of* $K_0(C(V) \rtimes_\alpha \mathbb{R})$ *by* $\hat{\mu}$, *is the group image of* $H^1(V, \mathbb{Z})$ *by the linear form* $[C]$.

Proof. By Corollary 1 and Appendix 3, we just have to compute the set of $(1/2i\pi)\ \mu_n(\delta_n(u)u^*)$, where δ_n is the action of the vector field X on the constant bundle with fiber $M_n(\mathbb{C})$, μ_n is the tensor product of μ by the ordinary trace on $M_n(\mathbb{C})$, and u is a smooth map of V in the unitary group $U(n)$. We thus get $(1/2i\pi)\ \mu_n(\delta_n(u)u^*) = (1/2i\pi) \int \operatorname{Trace}((Xu)u^*)\, d\mu$. For any $u \in C^\infty(V, U(n))$, the equality $(1/2i\pi) \operatorname{Trace}((du)u^*)$ defines a closed 1-form ω which is the pull back by the map $\det u$: $V \to U(1)$ of the fundamental class of $U(1)$. This shows that the cohomology class $[\omega] \in H^1(V, \mathbb{R})$ is the image of an element of $H^1(V, \mathbb{Z})$ and proves that $\langle H^1(V, \mathbb{Z}), [C]\rangle$ contains the image of $\hat{\mu}$. Conversely, given any closed 1-form ω, which is integral, there exists $u \in C^\infty(V, U(1))$ such that $\omega = (1/2i\pi)$ $(du)u^{-1}$ so one gets the other inclusion. Q.E.D.

COROLLARY 3. *Let* V *be a compact smooth manifold with* $H^1(V, \mathbb{Z}) = 0$, ϕ *a minimal diffeomorphism of* V, *then the crossed product of* $C(V)$ *by* ϕ^* *is a simple* C^* *algebra without nontrivial idempotent.*

Proof. By amenability of $\mathbb{Z}$, let μ be a ϕ-invariant measure on V, and $\hat{\mu}$ be the dual trace on $C(V) \rtimes_{\phi^*} \mathbb{Z}$. As ϕ is minimal, $C(V) \times_{\phi^*} \mathbb{Z}$ is simple [13], so $\hat{\mu}$ is faithful, and $\hat{\mu}(e) \in \{0, 1\}$, e projection, $\Rightarrow e \in \{0, 1\}$ hence it remains only to show that the range of $K_0\ (C(V) \times_{\phi^*} \mathbb{Z})$ by $\hat{\mu}$ is equal to $\mathbb{Z}$. To do that, we use the canonical flow X on the mapping torus of ϕ, defined as the

[4] τ is assumed semi-finite and semi-continuous, A not necessarily unital.

quotient W of $V \times \mathbb{R}$ by the diffeomorphism $\Psi(x, s) = (\phi(x), s+1)$, while the flow X is given by $\partial/\partial s$. Since $H^1(V, \mathbb{Z}) = 0$ and V is connected (ϕ is minimal), $H^1(W, \mathbb{Z})$ is equal to $\mathbb{Z}$ and the 1-form corresponding to the generator is $ds = \omega$. Let $\mu_W = \mu_V \times ds$ be the corresponding invariant measure for X. It is clear that $\langle \omega, C \rangle = 1$, where C is the Ruelle–Sullivan current of μ_W, since $\omega(\partial/\partial s)$ is the constant function 1. Thus by Corollary 2 we know that the range of $\hat{\mu}_W$ is $\mathbb{Z}$, it remains to show its equality with the range of $\hat{\mu}_V$. But the C^* dynamical system $(C(W), \mathbb{R}, X)$ is induced from $(C(V), \mathbb{Z}, \phi^*)$ in the sense of [8]. It follows that $C(W) \rtimes_X \mathbb{R}$ is strongly Morita equivalent to $C(V) \rtimes_{\phi^*} \mathbb{Z}$, the trace $\hat{\mu}_W$ corresponding to $\hat{\mu}_V$, hence the equality. Q.E.D.

EXAMPLE 4. Let ϕ be a minimal diffeomorphism of the 3 sphere S^3([7]), then, by Corollary 3, the crossed product $C(S^3) \rtimes_{\phi^*} \mathbb{Z}$ is a simple unital C^* algebra without nontrivial idempotent. See also [3] for another solution of this problem of I. Kaplansky. A more explicit example was pointed out to me by M. Herman and D. Sullivan.

EXAMPLE 5. Let Γ be a discrete cocompact subgroup of $SL(2, \mathbb{R})$ such that with $V = SL(2, \mathbb{R})/\Gamma$, one has $H^1(V, \mathbb{Z}) = 0$, then for any $t \neq 0$, the diffeomorphism ϕ_t of V defined by the horocycle flow: $\phi_t(g\Gamma) = b_t g\Gamma$, $b_t = \left[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right]$ is minimal.

As another application of Theorem IV we now derive the remarquable result of Pimsner and Voiculescu [12] for crossed products by an automorphism. Given a C^* algebra B and an automorphism θ of B, the induced C^* dynamical $\mathbb{R}$ system $(A, \mathbb{R}, \alpha)$ is defined as follows: A is the C^* algebra of all norm continuous maps from $\mathbb{R}$ to B such that $x(s+1) = \theta(x(s))$ $\forall s \in \mathbb{R}$, while the action α_t of $\mathbb{R}$ is defined by $(\alpha_t(x))(s) = x(s-t)$, $\forall s, t$.

COROLLARY 6. *$K_j(B \rtimes_\theta \mathbb{Z})$ is canonically isomorphic to $K_{j+1}(A)$.*

Proof. $B \rtimes_\theta \mathbb{Z}$ is strongly Morita equivalent to $A \rtimes_\alpha \mathbb{R}$ so the assertion follows from Theorem 2 of Section IV. Q.E.D.

To compute $K_j(A)$, we use the exact sequence:

$$0 \to SB \xrightarrow{i} A \xrightarrow{\sigma} B \to 0,$$

where for $x \in SB = C_0(]0, 1[) \otimes B$, $i(x)$ is the only element of A such that $i(x)(s) = x(s)$ $\forall s \in]0, 1[$, while for $x \in A$, $\sigma(x) = x(0) \in B$. The connecting maps for the corresponding six term exact sequence for K groups are easy to compute and one obtains the Pimsner–Voiculescu exact sequence:

$$\begin{tikzcd}
& K_0(B) \arrow[r, "\mathrm{id}-\theta_*^{-1}"] & K_0(B) \arrow[dr] & \\
K_1(B \rtimes_\theta \mathbb{Z}) \approx K_0(A) \arrow[ur] & & & K_1(A) \approx K_0(B \rtimes_\theta \mathbb{Z}) \arrow[dl] \\
& K_1(B) \arrow[ul] & K_1(B) \arrow[l, "\mathrm{id}-\theta_*^{-1}"'] &
\end{tikzcd}$$

To end this section we prove the following corollary of Theorem 2 of Section IV.

COROLLARY 7. *Let (A, G, α) be a C^* dynamical system where G is a simply connected solvable Lie group, let $j \in \mathbb{Z}/2$ be the* mod 2 *dimension of G, then $K_i(A \rtimes_\alpha G)$ is isomorphic with $K_{i+j}(A)$.*

Proof. By Lemma 3.6 [9, p. 521], G is the semi-direct product $G = G_1 \rtimes \mathbb{R}$ of a closed normal simply connected subgroup G_1 by the real line, thus the proof follows by induction. Q.E.D.

VI. APPENDIX

In this section we shall give the proof of several known simple facts needed in the text.

Let A be a C^* algebra, $\tilde{A}$ the associated unital C^* algebra, the split exact sequence $0 \to A \to^{i_A} \tilde{A} \to^{\varepsilon_A} \mathbb{C} \to 0$, with retraction r_A, $r_A(\lambda) = \lambda 1_{\tilde{A}}$, $\forall \lambda \in \mathbb{C}$ allows, when dealing with K theory to assume A unital. By definition, $K_0(A)$ is the kernel of $(\varepsilon_A)_*$: $K_0(\tilde{A}) \to K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(A) = K_1(\tilde{A})$ since $K_1(\mathbb{C}) = 0$. Both K_0 and K_1 are functors from the category of C^* algebras with, not necessarily unital, homomorphisms as morphisms, to the category of abelian groups. In fact, if ρ: $A \to B$ is not unital, then $\tilde{\rho}$: $\tilde{A} \to \tilde{B}$ is unital and $\varepsilon_B \circ \tilde{\rho} = \varepsilon_A$ so $\tilde{\rho}_*$ maps $K_i(A)$ to $K_i(B)$, $i \in \mathbb{Z}/2$.

We wish to discuss successively the notions of suspension, tensor product, smoothing, traces, and unbounded multipliers.

1. *Suspension*

For $a < b$, $a, b \in [-\infty, +\infty]$, let $I =]a, b[$, $I' =]a, b]$. Given a unital C^* algebra A, we let SA (resp. CA) the suspension of A (resp. the cone over A) be the C^* tensor product of $C_0(I)$ (resp. $C_0(I')$) by A. We identify $\widetilde{SA}$ and $\widetilde{CA}$ (resp. SA, CA) with the subalgebras of $C(\bar{I}) \otimes A$, the algebra of continuous maps from $\bar{I}$ to A, defined by

$$\widetilde{SA} = \{x, x(a) = x(b) \in \mathbb{C}\}, \qquad SA = \{x, x(a) = x(b) = 0\},$$

$$\widetilde{CA} = \{x, x(a) \in \mathbb{C}\}, \qquad CA = \{x, x(a) = 0\}.$$

The C^* algebra CA is contractible, and the Bott periodicity is equivalent to the bijectivity of the connecting map $s_A^0\colon K_0(A)\to K_1(SA)$, of the exact sequence:

$$0\to SA\to CA \xrightarrow{\sigma} A\to 0,$$

where $\sigma(x)=x(b)\ \forall x\in CA$.

For later use, let us recall the explicit simple form of the connecting maps s_A^j, $K_j(A)\to K_{j+1}(SA)$, $j\in\mathbb{Z}/2$, where we are allowed to assume that A is unital since $(Si_A)_*$ is always an injection of $K_{j+1}(SA)$ in $K_{j+1}(\widetilde{SA})$ (the exact sequence $0\to SA\to\widetilde{SA}\to C_0(I)\to 0$ is split).

LEMMA 1. (a) *Let f be a continuous function from $\bar I=[a,b]$ to $\mathbb{C}^*=\{z\in\mathbb{C},\ z\neq 0\}$, with $f(a)=f(b)=1$, and winding number 1, let $e\in \operatorname{Proj} M_n(A)$, then $s_A^0([e])$ is the class $[f\otimes e]$, of $1+(f-1)\otimes e\in GL_n(\widetilde{SA})$.*

(b) *Let $t\to R(t)$ be a continuous map of $\bar I$ to $U(2)\subset M_2(\mathbb{C})$ with $R(a)=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, $R(b)=\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$. For any $u\in U_n(A)$*[5] *one has $s_A^1([u])=[e]-[e_0]$, where e_0, e are the projections:*

$$e_0(t)=\begin{bmatrix}1&0\\0&0\end{bmatrix}\ \forall t\in\bar I,\qquad e(t)=W(t)\,e_0(t)\,W(t)^*,$$

$$W(t)=R(t)\begin{bmatrix}u^{-1}&0\\0&1\end{bmatrix}R(t)^*.$$

Proof. (a) Let $\theta\in C(\bar I)$ such that $f(t)=\exp 2\pi i\theta(t)\ \ \forall t\in\bar I$, while $\theta(a)=0$, $\theta(b)=1$, and $x(t)=\theta(t)\ e\in M_n(A)$, $\forall t\in\bar I$. Then $\sigma(x)=e$ while $x\in M_n(CA)$, so $s_A^0[e]$ is represented by $\exp(i2\pi x)$.

(b) Let $v(t)=W(t)\ \left[\begin{smallmatrix}u&0\\0&1\end{smallmatrix}\right]$, $\forall t\in\bar I$, then

$$v(a)=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ v(b)=\begin{bmatrix}u&0\\0&u^{-1}\end{bmatrix}$$

so $v\in M_2(M_n(\widetilde{CA}))$ and

$$\sigma(v)=\begin{bmatrix}u&0\\0&u^{-1}\end{bmatrix}.$$

So by definition of the connecting map s_A^1, one gets $s_A^1([u])=[e]-[e_0]$.

Q.E.D.

[5] The unitary group of $M_n(A)$.

2. *Tensor products*

Given C^* algebras A and B we let, for definiteness, $A \otimes B$ be their minimal tensor product. We want explicit formulas for the pairing $\boxtimes$: $K_i(A) \times K_j(B) \to K_{i+j}(A \otimes B)$ (cf. [1, p. 77] or [10, p. 272]). First the two split exact sequences:

$$0 \to A \otimes B \to \tilde{A} \otimes B \to B \to 0, \qquad 0 \to \tilde{A} \otimes B \to \tilde{A} \otimes \tilde{B} \xrightarrow{\sigma} D \to 0,$$

where $D = (A \times B)^{\sim}$, give an isomorphism of $K_i(A \otimes B)$ with the kernel of σ_* in $K_i(\tilde{A} \otimes \tilde{B})$ (note, however, that the sequence $0 \to A \otimes B \to \tilde{A} \otimes \tilde{B} \to^{\sigma} D \to 0$ is not split in general). This will allow us to assume that A and B are unital. For $e \in \operatorname{Proj} M_p(A)$, $f \in \operatorname{Proj} M_q(B)$, the external tensor product of their classes in K_0 is defined by $[e] \boxtimes [f] = [e \otimes f]$. For other values of $i, j \in \mathbb{Z}/2$, the pairing $\boxtimes$: $K_i(A) \times K_j(B) \to K_{i+j}(A \otimes B)$ is defined by the equalities:

$$[u] \boxtimes [e] = (s^1)^{-1}((s^1[u]) \boxtimes [e]), \qquad u \in U_p(A),$$
$$[u] \boxtimes [v] = (s^1 \circ s^0)^{-1}((s^1[u] \boxtimes (s^1[v]), \qquad u \in U_p(A), v \in U_q(B).$$

LEMMA 2. *Let $u \in U_p(A)$, $e \in \operatorname{Proj} M_q(B)$, then $[u] \boxtimes [e] = [u \otimes e]$ is the class of $1 \otimes (1 - e) + u \otimes e \in U_{pq}(A \otimes B)$.*

(b) *Let h be a continuous map of degree one of $T^2 = \{(z_1, z_2), z_i \in \mathbb{C}, |z_i| = 1\}$ in $P_1(\mathbb{C}) = \operatorname{Proj} M_2(\mathbb{C})$, then given $u \in U_p(A)$, $v \in U_q(B)$, one has $[u] \boxtimes [v] = [h(u, v)] - [e_0]$, where $e_0 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \in M_2(M_{pq}(A \times B))$ and $h(u, v)$ is the matrix with entries $h_{ij}(u \otimes 1, 1 \otimes v) \in M_p(A) \otimes M_q(B)$.*

Proof. (a) Follows from Lemma 1.

(b) By naturality of s^1, s^0 and of the construction (b) one can replace $M_p(A)$ and $M_q(B)$ by $C(T)$, and u, v by the canonical unitary $U(z) = z$ $\forall z \in T$ in $C(T)$; the conclusion follows then from the commutative case [1]

Q.E.D.

COROLLARY 3. *Let b (resp. β) be the canonical generator of $K_1(C_0(I))$ (resp. $\beta = b \boxtimes b \in K_0(C_0(I^2))$), then*:

(a) *s^i is the operation of external product by b,*

(b) *$s^i \circ s^{i+1}$ is the operation of external product by β.*

Proof. (a) By Lemmas 2a and 1a, $s^0([e]) = [b] \boxtimes [e]$ $\forall [e] \in K_0(A)$, by Lemmas 2b and 1b, $s^1([u]) = [b] \boxtimes [u]$ $\forall [u] \in K_1(A)$.

(b) Follows from the associativity of the external product. Q.E.D.

3. *Smoothing*

Let A be a C^* algebra, $\mathscr{A}$ a norm dense $*$ subalgebra with the following property:

$$\forall x \in M_n(\mathscr{A}),\ \forall f \text{ analytic on a neighborhood of } \operatorname{Sp}_A(x),$$

with $f(0) = 0$, one has $f(x) \in M_n(\mathscr{A})$.

The main examples we use are:

(a) Given a C^* dynamical system (A, G, α) with G a Lie group, the subalgebra $A^\infty = \{x \in A,\ g \to \alpha_g(x) \in A \text{ of class } C^\infty\}$.

(b) Given a semi-finite semi-continuous trace τ on A, the ideal Dom τ.

The proof of [1, p. 164] and the existence of the polar decomposition for elements of $GL_n(\tilde{A}^\infty)$, show that the injection of $U_n(\mathscr{A})$ in $U_n(A)$ defines an isomorphism of $\pi_i(U(\mathscr{A}))$ with $\pi_i(U(A))$, we want the analogous statement for Proj $M_n(A)$.

LEMMA 4. *Any $e \in$ Proj $M_n(A)$ is equivalent to an $e' \in$ Proj $M_n(\mathscr{A})$.*

(b) *$e_1, e_2 \in$ Proj $M_n(\mathscr{A})$ are equivalent in $M_n(A)$ iff they are already in $M_n(\mathscr{A})$.*

Proof. (a) Approximate $e = e^*$ by $x = x^* \in M_n(\mathscr{A})$, whose spectrum does not contain $\frac{1}{2}$ and use the analytic functional calculus to replace x by its spectral projection for $[\frac{1}{2}, +\infty[$, this projection f belongs to $M_n(\mathscr{A})$ and if x is close enough to e, $\|f - e\| < 1$ so that f is equivalent to e.

(b) $e_2 M_n(\mathscr{A}) e_1$ is contained in $M_n(\mathscr{A})$ and dense in $e_2 M_n(A) e_1$ so it contains u with u^*u invertible in $M_n(\mathscr{A}))_{e_1}$ uu^* in $(M_n(\mathscr{A}))_{e_2}$, so e_1 is equivalent to e_2 in $M_n(\mathscr{A})$. Q.E.D.

4. *Traces*

Let B be a (nonunital) C^* algebra, τ a semi-finite semi-continuous trace on B, Dom τ the ideal of definition of τ. Then τ defines an additive map, noted τ_*, from $K_0(B)$ to $\mathbb{R}$. The simplest way to see it is to define on $\tilde{\mathscr{B}}$, the algebra obtained from $\mathscr{B} = \text{Dom } \tau$ by adjoining a unit, a trace $\tilde{\tau}$ which coincides with τ on $\mathscr{B}$ and takes the value 0 on $\mathbb{C} \subset \tilde{\mathscr{B}}$. Of course $\tilde{\tau}$ is not positive, but putting $\tau_*([e]) = \tilde{\tau}_n(e)$ for any $e \in$ Proj $M_n(\tilde{\mathscr{B}})$ one gets an additive map from $K_0(\mathscr{B}) = K_0(B)$ (Appendix 3) to $\mathbb{R}$.

Another, less simple but more conceptual way, is to consider the homomorphism ρ from the C^* algebra B to the Breuer ideal J in the von Neumann algebra N associated to τ, then by the Breuer theory τ defines an additive map of $K_0(J)$ to $\mathbb{R}$, which when composed with ρ_* gives the above map $\tau_* : K_0(B) \to \mathbb{R}$.

Now, by Bott periodicity, $K_0(B)$ can be described by loops in $GL(\tilde{B})$, we wish to express the above map from $K_0(B)$ to $\mathbb{R}$ in terms of loops.

We turn Dom τ into a Banach space, with norm $\|x\| = \|x\|_B + \tau(|x|)$.

LEMMA 5. *Let $t \to \mathscr{C}(t)$ be a C^1 loop, $t \in \bar{I} = [0, 2\pi]$ with values in $GL_n(\widetilde{\mathrm{Dom}}\,\tau)$, and $[\mathscr{C}]$ the corresponding element of $K_0(B)$. Then*

$$\tau_*[\mathscr{C}] = \frac{1}{2i\pi} \int \tilde{\tau}(\mathscr{C}'(t)\,\mathscr{C}(t)^{-1})\,dt.$$

Proof. By [1, p. 164], $\pi_1(GL(\widetilde{\mathrm{Dom}}\,\tau)) = \pi_1(GL(\tilde{B}))$, so by Bott periodicity, $\mathscr{C}$ is stably homotopic in $GL(\widetilde{\mathrm{Dom}}\,\tau)$ to an elementary loop of the form $\mathscr{C}_0(t) = \exp\, ite\, \exp(-ite_0)$, where $e \in \mathrm{Proj}\, M_n(\widetilde{\mathrm{Dom}}\,\tau)$, $e_0 = \varepsilon_B(e) \in M_n(\mathbb{C}) \subset M_n(\widetilde{\mathrm{Dom}}\,\tau)$. So to check Lemma 5 one has to show that the right-hand side only depends on the homotopy class of $\mathscr{C}$, which follows from $\tilde{\tau}((\mathscr{C}_1\mathscr{C}_2)'(t)\mathscr{C}_1\mathscr{C}_2(t)^{-1}) = \tilde{\tau}(\mathscr{C}_1(t)'\;\mathscr{C}_1(t)^{-1}) + \tilde{\tau}(\mathscr{C}_2(t)'\;\mathscr{C}_2(t)^{-1})$ and from the existence, for $\|\mathscr{C}(t) - 1\| < 1$ of a suitable $\mathrm{Log}\,\mathscr{C}(t)$. Then one checks that the right side gives $\tilde{\tau}(e)$ for elementary loops. Q.E.D.

5. *Unbounded Multipliers*

The following notion is useful mainly for notational convenience. Let B be a C^* algebra without unit, a natural notion of selfadjoint unbounded multiplier H for B, is to give a continuous (for the multiplier topology) unitary representation π of $\mathbb{R}$ in $M(B)$. Given such a representation one gets a homomorphism π of $C^*(\mathbb{R}) = C_0(\hat{\mathbb{R}})$ in $M(B)$ and we would like to define H in such a way that for $f \in C_0(\hat{\mathbb{R}})$, $\pi(f) = f(H)$.

LEMMA 6. *Let T be a closed densely defined operator from the (Banach) space B in itself, such that:*

(1) *$T \pm \lambda i$ is invertible for λ real and large enough.*

(2) $y^*T(x) = T(y)^*x \;\forall x, y \in \mathrm{Dom}\, T$.

Then there exists a (unique) unitary representation of $\mathbb{R}$ in $M(B)$ so that $f(T)y = \pi(f)y \;\forall y \in B, f(z) = (z \pm \lambda i)^{-1} \;\forall z \in \mathbb{R}$.

Proof. One can assume that $\pm i \notin \mathrm{Sp}\, T$, let then $L = (T + i)^{-1}$, $R = {}^*(T - i)_*^{-1}$, let us show that the pair L, R of bounded operators in B defines a multiplier $z \in M(B)$. For this we just have to check that $xL(y) = R(x)y \;\forall x, y \in B$. Let $y = (T + i)y'$, $x^* = (T - i)x'$, one has then:

$$xL(y) = ((T - i)(x'))^*y' = x'^*(T + i)(y') \text{ (by 2)} = R(x)y.$$

So let $z \in M(B)$ with $L(x) = zx \;\forall x \in B$, and $R(x) = xz \;\forall x \in B$. Consider z as an element of the von Neumann algebra B^{**}. Then, as the map L and R have dense ranges in B it follows that the supports of z and of z^* are equal to 1 in B^{**}. So $S = z^{-1}$ makes sense as an unbounded operator affiliated to B^{**}, we want to see that $H = S - i$ is selfadjoint, i.e., since $S^* = (z^*)^{-1}$, that $z^* - z = 2iz^*z$. By construction, $(T+i)^{-1}(y) = zy$ $\forall y \in B$ and also: $(T-i)^{-1}y = R(y^*)^* = (y^*z) = z^*y$, $\forall y \in B$, thus the resolvent identity for T gives immediately the equality $z^* - z = 2iz^*z$. So $H = S - i$ is selfadjoint, $(H+i)^{-1}$ is bounded and equals $z \in M(B) \subset B^{**}$. It follows then that for any $f \in C_0(\mathbb{R})$, $f(H) \in M(B)$. Put $f_\varepsilon(t) = (t^2+1)^{-\varepsilon}$, then $f_\varepsilon(H) = (z^*z)^{2\varepsilon} \in M(B)$ and when $\varepsilon \to 0$ the argument of [11] shows that for any $y \in B$, one has $\|f_\varepsilon(H)y - y\| \to 0$. It follows then that $f(H) \in M(B)$ for any continuous bounded function on $\mathbb{R}$, since for $y \in B$ one has $f(H)u =$ norm limit of $f(H)(f_\varepsilon(H)y) = ff_\varepsilon(H)y \in B$. The same argument shows that for any $y \in B$ the map $t \in \mathbb{R} \to \exp(itH)y \in B$ is norm continuous, and it follows that this unitary representation of $\mathbb{R}$ in $M(B)$ satisfies the conditions of Lemma 6. Q.E.D.

DEFINITION 7. *A selfadjoint unbounded multiplier of B is an operator T satisfying the conditions of Lemma* 6.

Thus the notion is analogous to the classical notion of multipliers, one simply has to take care of the unboundedness problem.

COROLLARY 8. *If T is a selfadjoint unbounded multiplier and $P = P^* \in M(B)$, then $T + P$ is a selfadjoint unbounded multiplier.*

Proof. Condition 2 is clear for $T + P$, and condition 1 follows from the inequality $\|(H \pm \lambda i)^{-1}\| \leqslant |\lambda|^{-1}$ for $\lambda \in \mathbb{R}$ and the standard proof of selfadjointness for a bounded perturbation. Q.E.D.

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